

# Estimation of errors of quadrature formula for singular integrals of Cauchy type with special forms

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## Abstract

In this work, we consider the singular integrals of Cauchy type of the forms

$$J(f, x) = \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}(t-x)} dt, \quad -1 < x < 1. \quad (1)$$

and

$$\Phi(f, z) = -\frac{\sqrt{z^2-1}}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}(t-z)} dt, \quad z \notin [-1, 1]. \quad (2)$$

which are understood as Cauchy principal value integrals. Quadrature formulas (QFs) for singular integrals (SIs) (1) and (2) are of the forms

$$J(f, x) = \sum_{k=0}^N A_k(x) f(t_k) + R_N(f, x), \quad -1 < x < 1. \quad (3)$$

and

$$\Phi(f, z) = \sum_{k=0}^N B_k(z) f(t_k) + R_N^*(f, z), \quad z \notin [-1, 1]. \quad (4)$$

where  $z$  is complex variable with  $|Re(z)| > 1$ . With the help of linear spline interpolation, we have proved the rate of convergence of the errors of QFs (3) and (4) for different classes (i.e.  $H^\alpha([-1, 1], K)$ ,  $C^{m,\alpha}([-1, 1])$ ,  $W^r([-1, 1])$ ) of density function  $f(t)$ . It is shown that approximation by spline possesses more advantages than other kinds of approximation: it requires the minimum smoothness of density function  $f(x)$  to get good order of decreasing errors.

## 1 Introduction

The importance of singular integrals (SIs) of the form (1) and (2) and their numerical solution are given in many researchers work ([1]-[3]) and literatures cited therein. Many of them are based on the approximation of density function  $f(t)$  with Chebyshev polynomials.

Note that in (2), the function  $\sqrt{z^2-1}$  is understood as a single-valued branch in the plane of complex variable with cut along the interval  $[-1, 1]$  such that  $\sqrt{z^2-1} = z + O(z^{-1})$

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<sup>1</sup>This paper is dedicated to the memory of Professor Israilov M.I. (1934-2010). It was published in collection of papers entitled "Differential Equations and Inverse Problems, Press FAN, Tashkent, 1986, pp. 236-258 (in Russian)

for the large  $z > 0$ . In the future, under  $W = \arcsin z$  only a branch of the function for which  $|\operatorname{Re}(W)| < \frac{\pi}{2}$  will be understood.

In this paper, we construct efficient quadrature formulas (QFs) for SIs (1) and (2) using linear spline interpolation. Obtained QFs provide uniform convergence for any singular point  $x \in (-1, 1)$  and any  $z \notin [-1, 1]$ .

## 2 Construction of the quadrature formula

In order to write the exact form of coefficients of the quadrature formula (3) and (4), we introduce the following notations

$$G_k(x) = \left| \frac{t_k \sqrt{1-x^2} - x \sqrt{1-t_k^2}}{\sqrt{1-x^2} + \sqrt{1-t_k^2}} \right|,$$

$$g_k = \frac{1}{\pi h_k} (\arcsin t_{k+1} - \arcsin t_k),$$

$$F_k(z) = \frac{1}{\pi h_k} \left( \arcsin \frac{zt_{k+1} - 1}{z - t_{k+1}} - \arcsin \frac{zt_k - 1}{z - t_k} \right).$$

If  $x \neq t_k$ , then the coefficients of QFs (3) are computed by the formulas

$$\left. \begin{aligned} A_0(x) &= \frac{t_1 - x}{\pi h_0} \ln G_1(x) - \sqrt{1-x^2} g_0, \\ A_k(x) &= \frac{t_{k+1} - x}{\pi h_k} \ln \frac{G_{k+1}(x)}{G_k(x)} + \frac{x - t_{k-1}}{\pi h_{k-1}} \ln \frac{G_k(x)}{G_{k-1}(x)} \\ &\quad - \sqrt{1-x^2} (g_k - g_{k-1}), \quad k = 1, \dots, N-1 \\ A_N(x) &= \frac{t_{N-1} - x}{\pi h_{N-1}} \ln G_{N-1}(x) + \sqrt{1-x^2} g_{N-1}. \end{aligned} \right\}. \quad (5)$$

As  $G_k(\pm 1) = 1$  for all  $k$ , then  $A_k(\pm 1) = 0$  for  $k = 0, \dots, N$ . These correspond with the fact in [6] that  $J(f, x)|_{x=\pm 1} = 0$  is independent from the value of  $f(\pm 1)$ .

If  $x$  coincides with the nodes  $t_k$ , ( $k = 1, \dots, N-1$ ), then the coefficients  $A_j(t_k)$ ,  $j \neq k-1, k, k+1$  are computed by (5). If  $k$  in (5) is replaced by  $k-1$  and  $k+1$  and  $x = t_k$  is put, then coefficients  $A_{k-1}(t_k)$  and  $A_{k+1}(t_k)$  are again computed respectively by (5) and for  $A_k(t_k)$  we have

$$A_k(t_k) = \frac{1}{\pi} \ln \frac{G_{k+1}(t_k)}{G_{k-1}(t_k)} - \sqrt{1-t_k^2} (g_k - g_{k-1}), \quad k = 1, \dots, N-1. \quad (6)$$

Coefficients of the QFs (4) have the form

$$\left. \begin{aligned} B_0(z) &= (z - t_1)F_0(z) + \sqrt{z^2 - 1}g_0, \\ B_k(z) &= (z - t_{k+1})F_k(z) - (z - t_{k-1})F_{k-1}(z) \\ &\quad + \sqrt{z^2 - 1}(g_k - g_{k-1}), \quad k = 1, \dots, N-1 \\ B_N(z) &= (t_{N-1} - z)F_{N-1}(z) - \sqrt{z^2 - 1}g_{N-1}. \end{aligned} \right\}. \quad (7)$$

Let us derive coefficients of QFs which are given by (5) and (7). As we know the linear spline  $S_N(t)$  interpolating the given function  $f$  on the grid  $\Delta : -1 = t_0 < t_1 < \dots < t_{N-1} < t_N = 1$  for  $t \in [t_j, t_{j+1}]$  has the form

$$S_N(t) = \frac{1}{h_k} \left[ (t_{k+1} - t)f(t_k) + (t - t_k)f(t_{k+1}) \right] \quad (8)$$

Replacing  $f(t)$  in (1) with  $S_N(t)$  we have

$$\begin{aligned} J(S_N, x) &= \frac{\sqrt{1-x^2}}{\pi} \sum_{k=0}^{N-1} \frac{1}{h_k} \int_{t_k}^{t_{k+1}} \frac{[(t_{k+1} - t)f(t_k) + (t - t_k)f(t_{k+1})]}{\sqrt{1-t^2}(t-x)} dt. \\ &= \frac{\sqrt{1-x^2}}{\pi} \left[ \frac{1}{h_0} \int_{-1}^{t_1} \frac{(t_1 - t)dt}{\sqrt{1-t^2}(t-x)} f(t_0) + \frac{1}{h_{N-1}} \int_{t_{N-1}}^1 \frac{(t - t_{N-1})dt}{\sqrt{1-t^2}(t-x)} f(t_N) \right. \\ &\quad \left. + \sum_{k=1}^{N-2} \frac{1}{h_k} \left( \int_{t_k}^{t_{k+1}} \frac{(t - t_k)dt}{\sqrt{1-t^2}(t-x)} + \int_{t_{k+1}}^{t_{k+2}} \frac{(t_{k+2} - t)dt}{\sqrt{1-t^2}(t-x)} \right) f(t_{k+1}) \right]. \end{aligned} \quad (9)$$

Introducing notations

$$J(k, x) = \int_{t_k}^{t_{k+1}} \frac{dt}{\sqrt{1-t^2}(t-x)}, \quad J_1(k, x) = \int_{t_k}^{t_{k+1}} \frac{tdt}{\sqrt{1-t^2}(t-x)},$$

and using easy checking formulas

$$\begin{aligned} \int \frac{xdt}{\sqrt{1-t^2}(\sqrt{1-x^2} + \sqrt{1-t^2})} &= \ln \frac{1 + \sqrt{1-x^2}\sqrt{1-t^2} + xt}{\sqrt{1-x^2} + \sqrt{1-t^2}} + C, \\ \int \frac{tdt}{\sqrt{1-t^2}(\sqrt{1-x^2} + \sqrt{1-t^2})} &= -\ln(\sqrt{1-x^2} + \sqrt{1-t^2}) + C, \end{aligned}$$

obviously we have

$$J(k, x) = \frac{1}{1-x^2} \ln \left| \frac{(t-x)(1+\sqrt{1-x^2}\sqrt{1-t^2}+xt)}{(\sqrt{1-x^2}+\sqrt{1-t^2})^2} \right|_{t=t_k}^{t=t_{k+1}},$$

writing

$$(t-x)(1+\sqrt{1-x^2}\sqrt{1-t^2}+xt) = (\sqrt{1-x^2}+\sqrt{1-t^2})(t\sqrt{1-x^2}-x\sqrt{1-t^2}),$$

we obtain

$$J(k, x) = \frac{1}{1-x^2} \ln \left| \frac{t\sqrt{1-x^2}-x\sqrt{1-t^2}}{(\sqrt{1-x^2}+\sqrt{1-t^2})^2} \right|_{t=t_k}^{t=t_{k+1}} = \frac{1}{1-x^2} \ln \frac{G_{k+1}}{G_k(x)}. \quad (10)$$

Next, having the following relation

$$J_1(k, x) = xJ(k, x) + \pi h_k g_k. \quad (11)$$

Eq. (9) can be rewritten as

$$\begin{aligned} J(S_N, x) &= \frac{\sqrt{1-x^2}}{\pi} \left\{ \frac{1}{h_0} [t_1 J(0, x) - J_1(0, x)] f(-1) \right. \\ &\quad + \frac{1}{h_{N-1}} [J_1(N-1, x) - t_{N-1} J_1(N-1, x)] f(1) \\ &\quad + \sum_{k=1}^{N-1} \left[ \frac{1}{h_k} (t_{k+1} J(k, x) - J_1(k-1, x)) \right. \\ &\quad \left. \left. + \frac{1}{h_{k-1}} (J_1(k-1, x) - t_{k-1} J(k-1, x)) \right] f(t_k) \right\} \end{aligned} \quad (12)$$

Substituting (10) and (11) into (12) and simplifying the expressions, we arrive at (5) for finding the coefficients  $A_k(x)$  of the QFs (3). Furthermore, we can derive the Eq. (6) from (12) and (10)-(11) as follows

$$\begin{aligned} A_k(t_k) &= \frac{\sqrt{1-x^2}}{\pi} \left\{ \frac{1}{h_k} (t_{k+1} J(k, t_{k+1}) - J_1(k, t_k)) \right. \\ &\quad \left. + \frac{1}{h_{k-1}} (J_1(k-1, t_k) - t_{k-1} J(k-1, t_k)) \right\} \\ &= \frac{\sqrt{1-x^2}}{\pi} [J(k, t_k) - J(k-1, t_k) - \pi(g_k - g_{k-1})] \\ &= \frac{1}{\pi} \ln \frac{G_{k+1}(t_k)}{G_{k-1}(t_k)} - \sqrt{1-t_k^2} (g_k - g_{k-1}), \quad k = 1, \dots, N-1. \end{aligned} \quad (13)$$

In order to derive the coefficients  $B_k$  of the QFs (4) we use the combination of the integrals

$$J^*(k, z) = \int_{t_k}^{t_{k+1}} \frac{dt}{\sqrt{1-t^2}(t-z)}, \quad J_1^*(k, z) = \int_{t_k}^{t_{k+1}} \frac{t dt}{\sqrt{1-t^2}(t-z)}. \quad (14)$$

First  $J^*(k, z)$  is computed at  $z = x$ , where  $x$  is any number such that  $|x| > 1$ . Then continuing analytical function  $J^*(k, x)$  along the intervals  $(-\infty, -1)$ ,  $(1, \infty)$  on the plane of complex variable  $z$  with cut along the interval  $[-1, 1]$ , we obtain

$$J^*(k, z) = \frac{1}{\sqrt{z^2-1}} \arcsin \frac{zt-1}{z-t} \Big|_{t=t_k}^{t=t_{k+1}} = \frac{\pi h_k}{\sqrt{z^2-1}} F_k(z). \quad (15)$$

For  $J_1^*$

$$J_1^*(k, z) = zJ^*(k, z) + g_k \pi h_k. \quad (16)$$

Replacing  $x$  into  $z$  in (12) and using (14)-(16), we obtain the coefficients  $B_k$  of the QFs (4)

### 3 Estimation of errors

Let us introduce the following classes of functions:

1.  $H^\alpha([-1, 1], K)$  is a class function satisfying Holder condition on the interval with the index  $\alpha$  and constant  $K$ .
2.  $C^{m,\alpha}[-1, 1] = \left\{ f(t) : f^{(m)} \in H^\alpha([-1, 1], K_m) \right\}$
3.  $CC_\Delta^{m,\alpha}[-1, 1] = \left\{ f(t) : f(t) \in C[-1, 1] \text{ and } f(t) \in C^{m,\alpha}[t_k, t_{k+1}] \right\}$ .
4.  $W^r[-1, 1] = \left\{ f(t) : f^{(r-1)}(t) \text{ is absolutely continuous and } \operatorname{ess\,sup}_{|t| \leq 1} |f^{(r)}| = M_r \right\}$ .
5.  $CW_\Delta^r[-1, 1] = \left\{ f(t) : f(t) \in C[-1, 1] \text{ and } f(t) \in W^r[t_k, t_{k+1}] \right\}$ .

Everywhere we use the notation  $\|f\|_C = \|f(t)\|_{C[-1,1]}$  as a norm of the function. Note that  $M_r = \|f^{(r)}\|_C$  for any  $f(t) \in C^r[-1, 1]$ .

Now we prove the following theorems with respect to QFs (3) and (4).

**Theorem 1** *Let  $f(t)$  be a function belonging to one of the classes of functions  $W^1[-1, 1]$ ,  $CC_\Delta^{1,\alpha}$  or  $CW_\Delta^2[-1, 1]$ . Then for the errors of QFs (3) the estimations*

$$\|R_N(f, x)\|_C \leq L \frac{\ln N}{N^\beta},$$

*are true for all  $x \in (-1, 1)$ , where  $L$  and  $\beta$  are given in the Table 1.*

Table 1: For QFs (3)

Classes of functions	$\beta$	L
$W^1[-1, 1]$	1	$\frac{4\gamma M_1}{\pi} \left(1 + \frac{\pi\sqrt{2}}{2\gamma \ln N}\right)$
$CC_{\Delta}^{1,\alpha}[-1, 1]$	$1 + \alpha$	$\frac{2\gamma^{1+\alpha} K_1}{\pi} \left(1 + \frac{\pi\sqrt{2}}{2\gamma \ln N}\right)$
$CW_{\Delta}^2[-1, 1]$	2	$\frac{\gamma^2 M_2}{\pi} \left(1 + \frac{\pi\sqrt{2}}{\gamma \ln N}\right)$

Remark 1: In the case of uniform grids,  $\gamma = 2$ .

**Theorem 2** *Let  $f(t)$  satisfy the conditions of Theorem 1. Then the errors of QFs (4) are*

$$\max_z |R_N^*(f, z)| \leq L^* \frac{LnN}{N^\beta}, \quad L^* = \sqrt{L^2 + \left(\frac{L_1}{\ln N}\right)^2}$$

where  $L$ ,  $L_1$  and  $\beta$  are given in the Table 2.

Table 2: For QFs (4)

Classes of functions	$\beta$	L	$L_1$
$W^1[-1, 1]$	1	$\frac{4\gamma M_1}{\pi} \left(1 + \frac{\pi\sqrt{2}}{2\gamma \ln N}\right)$	$\frac{M_1 \gamma}{2\pi}$
$CC_{\Delta}^{1,\alpha}[-1, 1]$	$1 + \alpha$	$\frac{2\gamma^{1+\alpha} K_1}{\pi} \left(1 + \frac{\pi\sqrt{2}}{2\gamma \ln N}\right)$	$\frac{K_1 \gamma^{1+\alpha}}{4\pi}$
$CW_{\Delta}^2[-1, 1]$	2	$\frac{\gamma^2 M_2}{\pi} \left(1 + \frac{\pi\sqrt{2}}{\gamma \ln N}\right)$	$\frac{M_2 \gamma^2}{8\pi}$

Next theorem is again related to QFs (3) but in different classes of functions:

**Theorem 3** *Let  $f(t)$  be a function belonging to one of the classes of functions*

$$H^\alpha([-1, 1], K), \quad W^1[-1, 1], \quad CC_{\Delta}^{1,\alpha} \text{ or } CW_{\Delta}^2[-1, 1].$$

*Then the error terms of QFs (3) satisfy the following estimations*

$$\|R_N(f, x)\|_C \leq L_2 \frac{\ln N}{N^\beta},$$

for all  $x \in (-1, 1)$ , where  $L_2$  and  $\beta$  are given in the Table 3.

Remark 2: Note that the main terms of  $L_2$  in the Theorem 3 for the last three classes of functions is twice less than the main terms of  $L$  in the Theorem 1.

Table 3: For QFs (3)

Classes of functions	$\beta$	$L_2$
$H^\alpha([-1, 1], K)$	$\alpha$	$\frac{2^{2-\alpha}\gamma^\alpha K}{\pi} \left( 1 + \left( 2 + \frac{1}{\alpha} \right) \frac{2^{2-2\alpha}}{\gamma^\alpha \ln N} \right)$
$W^1[-1, 1]$	1	$\frac{2\gamma M_1}{\pi} \left( 1 + \frac{12\pi}{\gamma \ln N} \right)$
$CC_\Delta^{1,\alpha}[-1, 1]$	$1 + \alpha$	$\frac{\gamma^{1+\alpha} K_1}{\pi} \left( 1 + \frac{12\pi}{\gamma \ln N} \right)$
$CW_\Delta^2[-1, 1]$	2	$\frac{\gamma^2 M_2}{2\pi} \left( 1 + \frac{\pi \sqrt{24\pi}}{\gamma \ln N} \right)$

In estimation of the error of QFs (3) we use the idea of [4] (see also [3]) and the following Lemmas.

**Lemma 1** *Let  $S_N(t)$  be linear spline (8) interpolating  $f(t)$  on the grid  $\Delta$ , and let  $t \in [t_k, t_{k+1}]$ . Then for the estimate of error  $r_N(f, t) = S_N(t) - f(t)$ , we obtain*

$$\|r_N(f, t)\|_C = r_N^*(h_k),$$

where  $r_N^*(h_k)$  is defined by the Table (4)

Table 4: Error of the linear spline (8)

Classes of functions	$r_N^*(h_k)$
$H^\alpha([-1, 1], K)$	$\frac{1}{2^\alpha} K h_k^\alpha$
$W^1[-1, 1]$	$\frac{1}{2} h_k$
$CC_\Delta^{1,\alpha}[-1, 1]$	$\frac{1}{4} K_1 h_k^{1+\alpha}$
$CW_\Delta^2[-1, 1]$	$\frac{1}{8} h_k^2$

Lemma 1 is proved as Theorem 2.1 which is shown in [4].

**Lemma 2** *Let  $S_N(t)$  be linear spline defined by (8). Then*

$$r_N(f, t) \in H^1([-1, 1], \tilde{K}),$$

where  $\tilde{K}$  is given in Table (5).

Table 5: Error of the linear spline (8)

Classes of functions	$K$
$W^1[-1, 1]$	$2M_1$
$CC_{\Delta}^{1,\alpha}[-1, 1]$	$K_1 h_k^\alpha$
$CW_{\Delta}^2[-1, 1]$	$M_2 h_k$

**Proof of the Lemma 2.** Consider three cases:

$$(a) t, t' \in [t_k, t_{k+1}], \quad (b) |t - t'| \geq h_k \quad (c) \tau \in [t_{k-1}, t_k], \quad t' \in [t_k, t_{k+1}], \quad |t - t'| \leq h_k$$

**I.** Let  $f(t) \in W^1[-1, 1]$ . Then in the case (a), using representation (8) we have

$$\begin{aligned} \left| r_N(f; t) - r_N(f; t') \right| &= \frac{1}{h_k} \left| (t - t') [f(t_{k+1}) - f(t_k)] - h_k [f(t) - f(t')] \right| \\ &= \frac{1}{h_k} \left| (t - t') \int_{t_k}^{t_{k+1}} f'(s) ds - h_k \int_{t'}^t f'(s) ds \right| \leq 2M_1 |t - t'|. \end{aligned}$$

In the case (b), in accordance with Lemma 1, we get

$$\left| r_N(f; t) - r_N(f; t') \right| \leq \left| r_N(f; t) \right| + \left| r_N(f; t') \right| \leq h_k M_1 \leq M_1 |t - t'|.$$

Using the case (a), in the case (c) we obtain

$$\begin{aligned} \left| r_N(f; t) - r_N(f; t') \right| &\leq \left| r_N(f; t) - r_N(f; t_k) \right| + \left| r_N(f; t_k) - r_N(f; t') \right| \\ &\leq 2M_1 |t_k - t| + 2M_1 |t - t'| = 2M_1 |t - t'|. \end{aligned}$$

**II.** Now let  $f(t) \in CC_{\Delta}^{1,\alpha}[-1, 1]$ . In the case (a)

$$\begin{aligned} \left| r_N(f; t) - r_N(f; t') \right| &\leq \frac{1}{h_k} \left| (t - t') [f(t_{k+1}) - f(t_k)] - h_k [f(t) - f(t')] \right| \\ &= \frac{1}{h_k} \left| (t - t') (t_{k+1} - t_k) f'(\theta_1) - h_k (t - t') f'(\theta_2) \right| \\ &= \left| (t - t') \right| \left| f'(\theta_1) - f'(\theta_2) \right| \leq K_1 |t - t'| |\theta_1 - \theta_2|^\alpha \\ &\leq K_1 h_k^\alpha |t - t'|. \end{aligned}$$

In the case (b), due to Lemma 1, we have

$$\left| r_N(f; t) - r_N(f; t') \right| \leq K_1 h_k^\alpha |t - t'|.$$



It is obvious in the case (c) that

$$\left| r_N(f; t) - r_N(f; t') \right| \leq K_1 h_k^\alpha |t - t'|.$$

**III.** Let  $f(t) \in CW_{\Delta}^2[-1, 1]$ . In case (a), we have

$$\begin{aligned} \left| r_N(f; t) - r_N(f; t') \right| &= |t - t'| \left| f'(\theta_1) - f'(\theta_2) \right| \\ &= |t - t'| \left| \int_{\theta_1}^{\theta_2} f''(s) ds \right| \leq M_2 h_k |t - t'|. \end{aligned}$$

The cases (b) and (c) are proved in a similar way as the case (a). So that the proof of the Lemma 2 follows from the above obtained errors.

Now it is easy to prove the following lemma.

**Lemma 3** *Let  $S_N(t)$  be linear spline defined by (8) and  $f(t) \in H^\alpha([-1, 1], K)$ . Then*

$$r_N(f, t) \in H^\alpha([-1, 1], 2^{2-\alpha} K).$$

**Prove of the Theorem 1.** Since

$$\int_{-1}^1 \frac{dt}{\sqrt{1-x^2}(t-x)} = 0,$$

the reminder term of QFs (3) can be represented as

$$R_N(f, x) = \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^1 \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1-x^2}(t-x)} dt. \quad (17)$$

For definiteness, let us prove the Theorem 1 in case  $0 \leq x \leq 1$  (the case  $-1 \leq x \leq 0$  is considered analogically). Fixing the number  $0 < \delta_N < \frac{1}{2}$  and dividing the integral in (17) into three parts to yield

$$\begin{aligned} R_N(f, x) &= \frac{\sqrt{1-x^2}}{\pi} \left( \int_{-1}^{x-\delta_N} + \int_{x-\delta_N}^{x+\delta_N} + \int_{x+\delta_N}^1 \right) \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1-x^2}(t-x)} dt \\ &= \frac{\sqrt{1-x^2}}{\pi} (J_1 + J_2 + J_3). \end{aligned} \quad (18)$$

First assume that  $\delta_N < 1 - x$ . Then due to (10), for  $J_1$  we have

$$\begin{aligned} |J_1| &= 2\|r_N(f, x)\|_C \left| \int_{-1}^{x-\delta_N} \frac{dt}{\sqrt{1-x^2}(t-x)} \right| \\ &= \frac{2}{\sqrt{1-x^2}} \|r_N(f, x)\|_C \ln \left| \frac{t\sqrt{1-x^2} - x\sqrt{1-t^2}}{\sqrt{1-x^2} + \sqrt{1-t^2}} \right|_{t=x-\delta_N}. \end{aligned}$$

It is not hard to show that

$$\begin{aligned} \varphi_1(x, \delta_N) &= \frac{t\sqrt{1-x^2} - x\sqrt{1-t^2}}{\sqrt{1-x^2} + \sqrt{1-t^2}} \Big|_{t=x-\delta_N} \\ &= x + \frac{1}{\delta_N} \left( 1 - x^2 - \sqrt{1-x^2} \sqrt{1-(x-\delta_N)^2} \right). \end{aligned}$$

This is a function of  $x$  which strictly decreases on  $[0, \frac{\delta_N}{2}]$  and strictly increases on  $[\frac{\delta_N}{2}, 1]$ , and

$$\varphi_1(0, \delta_N) = \frac{\delta_N}{1 + \sqrt{1-\delta_N^2}}, \quad \varphi_1(\delta_N/2, \delta_N) = \frac{\delta_N}{2}, \quad \varphi_1(1, \delta_N) = 1.$$

Hence,

$$|J_1| \leq \frac{2}{\sqrt{1-x^2}} \|r_N(f, x)\|_C \ln \frac{2}{\delta_N}. \quad (19)$$

For  $J_2$ , we use Lemma 2

$$|J_2| \leq \tilde{K} \int_{x-\delta_N}^{x+\delta_N} \frac{dt}{\sqrt{1-t^2}} = \tilde{K} [\arcsin(x + \delta_N) - \arcsin(x - \delta_N)].$$

Let

$$\varphi_2(x, \delta_N) = \arcsin(x + \delta_N) - \arcsin(x - \delta_N).$$

Since  $0 \leq x \leq 1 - \delta_N$  and by assumption  $\delta_N \leq 1 - x$ , derivative of  $\varphi_2(x, \delta_N)$  is positive and

$$\varphi_2(x, \delta_N) \leq \varphi_2(1 - \delta_N, \delta_N) = \arcsin 2\sqrt{\delta_N(1 - \delta_N)}.$$

From this and the known inequality  $\arcsin \alpha \leq \frac{\pi}{2}\alpha$ , ( $0 \leq \alpha \leq \frac{\pi}{2}$ ) it follows that

$$\varphi_2(x, \delta_N) \leq \pi\sqrt{\delta_N}.$$

Hence

$$|J_2| \leq \tilde{K}\pi\sqrt{\delta_N} \quad (20)$$

For  $J_3$ , we have

$$\begin{aligned} |J_3| &= 2 \|r_N(f, x)\|_C \left| \int_{x+\delta_N}^1 \frac{dt}{\sqrt{1-x^2}(t-x)} \right| \\ &= \frac{2}{\sqrt{1-x^2}} \|r_N(f, x)\|_C (-1) \ln \left| \frac{t\sqrt{1-x^2} - x\sqrt{1-t^2}}{\sqrt{1-x^2} + \sqrt{1-t^2}} \right|_{t=x+\delta_N}. \end{aligned}$$

We may show that the function

$$\varphi_3(x, \delta_N) = \frac{t\sqrt{1-x^2} - x\sqrt{1-t^2}}{\sqrt{1-x^2} + \sqrt{1-t^2}} \Big|_{t=x+\delta_N} = -\varphi_1(x, -\delta_N),$$

strictly increases on  $[0, 1 - \delta_N]$  and strictly decreases from  $\frac{\delta_N}{1+\sqrt{1-\delta_N^2}}$  to 1. So that

$$|J_3| \leq \frac{2}{\sqrt{1-x^2}} \|r_N(f, x)\|_C \ln \frac{2}{\delta_N}. \quad (21)$$

It follows from the errors of (19)- (21) and (24) that

$$\|R_N(f, x)\|_C \leq \frac{4}{\pi} \|r_N(f, x)\|_C \ln \frac{2}{\delta_N} + \tilde{K} \sqrt{\delta_N}. \quad (22)$$

Now consider the case  $\delta_N > 1 - x$ . Write

$$\begin{aligned} R_N(f, x) &= \frac{\sqrt{1-x^2}}{\pi} \left( \int_{-1}^{x-\delta_N} + \int_{x-\delta_N}^1 \right) \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1-x^2}(t-x)} dt \\ &= \frac{\sqrt{1-x^2}}{\pi} (J_1^* + J_2^*) \end{aligned} \quad (23)$$

Integral  $J_1^*$  is estimated as  $J_1$ . Due to Lemma 2

$$|J_2^*| \leq \tilde{K} \int_{x-\delta_N}^1 \frac{dt}{\sqrt{1-t^2}} = \tilde{K} \arcsin \sqrt{1-(x-\delta_N)^2}.$$

Since  $0 < x \leq 1$ ,  $\delta_N > 1 - x$  and due to the inequality

$$1 - (x - \delta_N)^2 = (1 - x + \delta_N)(1 + x - \delta_N) < 4\delta_N.$$

we obtain

$$|J_2^*| \leq \tilde{K} \arcsin 2\sqrt{\delta_N} \leq \tilde{K} \pi \sqrt{\delta_N}.$$

Substituting the errors of  $J_1^*$  and  $J_2^*$  into (23), we arrive at estimation (22).

In order to determine the errors of estimation for every classes of functions in Theorem 1, we use the results of Lemma 1 and 2 and set  $\delta_N = \frac{2}{N^2}$ . Viz:

**I.** Let  $f(t) \in W^1[-1, 1]$ . Then

$$\begin{aligned} \|R_N(f, x)\|_C &\leq \frac{2}{\pi} M_1 h \ln \frac{2}{\delta_N} + 2M_1 \sqrt{\delta_N} \\ &= \frac{4M_1 \gamma}{\pi} \left( 1 + \frac{\pi \sqrt{2}}{2\gamma \ln N} \right) \frac{\ln N}{N}. \end{aligned}$$

**II.** If  $f(t) \in CC_{\Delta}^{1,\alpha}[-1, 1]$ , then

$$\begin{aligned} \|R_N(f, x)\|_C &\leq \frac{K_1}{\pi} h^{1+\alpha} \ln \frac{2}{\delta_N} + 2K_1 h^\alpha \sqrt{\delta_N} \\ &= \frac{2K_1 \gamma^{1+\alpha}}{\pi} \left( 1 + \frac{\pi \sqrt{2}}{2\gamma \ln N} \right) \frac{\ln N}{N^{1+\alpha}}. \end{aligned}$$

**III.** If  $f(t) \in CW_{\Delta}^2[-1, 1]$ , then

$$\begin{aligned} \|R_N(f, x)\|_C &\leq \frac{M_2}{2\pi} \frac{\gamma^2}{N^2} \ln \frac{2}{\delta_N} + \frac{2M_2 \gamma}{N} \sqrt{\delta_N} \\ &= \frac{M_2 \gamma^2}{\pi} \left( 1 + \frac{\pi \sqrt{2}}{\gamma \ln N} \right) \frac{\ln N}{N^2}. \end{aligned}$$

Theorem 1 is proved.

**Proof of the Theorem 2** is carried out by the famous scheme of the formula Sokhotskii-Plemergh (see [5]), principle maximum module for analytical function and results of Theorem 1 and Lemma 1.

**Proof of the Theorem 3.** Let the remainder term of QFs (3) be divided into three parts

$$\begin{aligned} R_N(f, x) &= \frac{\sqrt{1-x^2}}{\pi} \left( \int_{-1}^{x-\delta_N} + \int_{x-\delta_N}^{x+\delta_N} + \int_{x+\delta_N}^1 \right) \frac{r_N(f, t) - r_N(f, x)}{\sqrt{1-x^2}(t-x)} dt \\ &= \frac{\sqrt{1-x^2}}{\pi} (\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3). \end{aligned} \tag{24}$$

In the proof of Theorem 1, we have already seen that the case  $\delta_N < 1 - x$ , is adequate for the estimations of  $\tilde{J}_1$  and  $\tilde{J}_3$  i.e.

$$|\tilde{J}_1| + |\tilde{J}_3| \leq \frac{4}{\pi} \sqrt{1-x^2} \|r_N(f, x)\|_C \ln \frac{2}{\delta_N}. \tag{25}$$

For the estimation of  $\tilde{J}_2$ , we consider the function

$$T(x, \varepsilon, \sigma) = \sqrt{1-x^2} \int_{x-\varepsilon}^{x+\sigma} \frac{|t-x|^{\alpha-1}}{\sqrt{1-t^2}} dt, \tag{26}$$

where  $0 < \sigma \leq 1$ ,  $1 - x \geq \varepsilon$ ,  $0 < \varepsilon < \frac{1}{2}$ .

It is obvious that

$$\begin{aligned}
 T(x, \varepsilon, \sigma) &\leq 2\varepsilon^\sigma \int_0^1 \frac{\sqrt{1-x^2}y^{\sigma-1}}{\sqrt{1-(x+\varepsilon y)^2}} dy \\
 &= 2\varepsilon^\sigma \left[ \int_0^{1/2} \frac{\sqrt{1-x^2}y^{\sigma-1}}{\sqrt{1-(x+\varepsilon y)^2}} dy + \int_{1/2}^1 \frac{\sqrt{1-x^2}y^{\sigma-1}}{\sqrt{1-(x+\varepsilon y)^2}} dy \right] \\
 &= 2\varepsilon^\sigma [T_1(x, \varepsilon, \sigma) + T_2(x, \varepsilon, \sigma)].
 \end{aligned} \tag{27}$$

Let  $k_0 = [1/\varepsilon]$ , since  $1 - x \geq \varepsilon$ , then for some  $k, 1 \leq k \leq k_0$  the inequality

$$k\varepsilon \leq 1 - x < (k+1)\varepsilon,$$

takes place. From this and  $0 < y \leq \frac{1}{2}$  it follows that

$$\frac{1-x^2}{1-(x+\varepsilon y)^2} = \frac{(1-x)(1+x)}{(1-x-\varepsilon y)(1+x+\varepsilon y)} \leq \frac{1-x}{1-x-\varepsilon/2} \leq \frac{k+1}{k-1/2} \leq 4,$$

for all  $k \geq 1$ . Hence

$$T_1(x, \varepsilon, \sigma) = 2 \int_0^{1/2} y^{\sigma-1} dy = \frac{2^{1-\sigma}}{\sigma}. \tag{28}$$

Furthermore

$$T_2(x, \varepsilon, \sigma) = \frac{1}{2^{\sigma-1}} \int_{1/2}^1 \frac{\sqrt{1-x^2}}{\sqrt{1-(x+\varepsilon y)^2}} dy.$$

Let  $\varepsilon \leq 1 - x < 2\varepsilon$ . Then

$$\frac{1-x^2}{1-(x+\varepsilon y)^2} \leq \frac{1-x}{1-x-\frac{\varepsilon}{2}} \leq \frac{2}{1-y},$$

and therefore

$$T_2(x, \varepsilon, \sigma) = 2^{1-\sigma} \sqrt{2} \int_{1/2}^1 (1-y)^{1/2} dy = 2^{2-\sigma}. \tag{29}$$

If for some  $k \leq 2$  the inequality

$$k\varepsilon \leq 1 - x \leq (k+1)\varepsilon$$

takes place, then

$$\frac{1-x^2}{1-(x+\varepsilon y)^2} \leq \frac{1-x}{1-x-\varepsilon y} \leq \frac{k+1}{k-1} \leq 3,$$

therefore

$$T_2(x, \varepsilon, \sigma) \leq 2^{1-\sigma} \sqrt{3} \cdot \frac{1}{2} < 2^{2-\sigma}. \quad (30)$$

From (27)-(30) it follows that

$$T(x, \varepsilon, \sigma) \leq 2^{1-\sigma} \left( 2 + \frac{1}{\sigma} \right) \varepsilon^\sigma. \quad (31)$$

Now for  $\tilde{J}_2$  we have

$$\tilde{J}_2 \leq \frac{K}{\pi} \sqrt{1-x^2} \int_{x-\delta_N}^{x+\delta_N} \frac{|t-x|^{\sigma-1}}{\sqrt{1-t^2}} dt = \frac{K}{\pi} T(x, \varepsilon, \sigma), \quad (32)$$

where  $\sigma = \alpha$  for the class  $H^\alpha([-1, 1], K)$  and  $\sigma = 1$  for rest classes of functions. Assuming  $\sigma = \frac{2}{N}$  and from the Lemmas 2, 3 and inequalities (25) and (31)-(32) we get the assertion of the Theorem 3.

## References

- [1] S. Belotserkovskii, I. Lifanov (1985) *Numerical methods in singular integral equations. Moscow: Nauka, 254 p. (in Russian).*
- [2] Gabdul Khaev B.G. Finite-dimensional approximation of singular integrals and direct methods for special integrals and integra-differential equations. Mathematical analysis. Resume science and technology. VINITI AN SSSR, V.18, pp. 251-307.
- [3] Gabdul Khaev B.G. Optimal approximation for the solution of linear problems. Kazan: Kazan University press, 1980. 231 p.
- [4] Makavoz Yu. I., Sheshko M.A. On a error of estimation of quadrature formulas for singular integrals. Isv. AN SSSR. Ser. Phiz-Math. Nauk. 1977. V.6, pp. 36-41.
- [5] Muskhelishvili N.I. (1953) *Singular Integral equations. Gostekhizda (1946). M: Fizmatgiz, 1962. 512p.*
- [6] Pikhtiev G.H. Accurate methods for evaluation of Cauchy type singular integrals. Novosibirsk: Science, 1980.
- [7] U.S.Zavvalov, B.I.Kvasov, B.I.Miroshnichenko. Methods of spline functions, Nauka, Moskov, 1980